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Note on the Central Limit Theorem

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Note on the Central Limit Theorem

by
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§1. Introduction. The well-known theorem of Lindeberg asserts that for a sequence $X_1, X_2, \ldots$ of mutually independent random variables such that the mean of each X_k is 0, and each X_k has finite variance, in order that

$$(1.1) \qquad F_n^*(s_n x) \longrightarrow \Phi(x) \qquad \text{for all } x;$$

and

$$(1.2) \qquad F_k(s_n x) \longrightarrow \varepsilon(x) \qquad \text{for all } x \neq 0 ,$$

uniformly in k, for $1 \leq k \leq n$; (here $\Phi(x)$ is the standard normal distribution with mean 0 and variance 1; $\varepsilon(x)$ is the unit distribution with mean zero and variance 0; $F_k(x)$ is the distribution function of X_k, $F_n^*(x)$ is the distribution function of $S_n = X_1+\ldots+X_n$, and s_n^2 is the variance of S_n) it is necessary and sufficient that for every $\eta > 0$,

$$(1.3) \qquad \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x| \leq \eta s_n} x^2 dF_k = 1 .$$

The condition (1.3) is usually referred to as the "Lindeberg condition".

In the book of Gnedenko and Kolmogoroff, Limit Distributions for Sums of Independent Random Variables, (p. 5), it is implied that the condition

$$(1.4) \qquad \sum_{k\leq n} P(|X_k| > \eta s_n) \longrightarrow 0 ,$$

which is clearly a consequence of (1.3), is in fact equivalent to (1.3), in the setting given above. This (as we shall show) is not true, and the question arises as to just what is implied by (1.4) concerning the possible limit distributions of the partial sums. We propose to answer this question in this note.

We call $G(x)$ an accumulation d.f. if for some sequence $n_i \to \infty$, the normalized partial sums S_{n_i} have d.f.'s which converge to $G(x)$. The following is the theorem which will be provided in answer to the question raised above.

Theorem: If we assume that $s_n \to \infty$ as $n \to \infty$, and that (1.4) holds for all $\eta > 0$, then the accumulation distributions of the partial sums form a one parameter family of normal distributions. This parameter is the variance of the accumulation distribution, and ranges over a closed subinterval of $[0,1]$. This closed subinterval is in fact the set of limit points of $L(n)$.*

An example will be given in which the parameter range is the entire interval $[0,1]$. Similar examples may be constructed in which the parameter range is any given closed subinterval of $[0,1]$.

§2. Proof of the Theorem. The assumption of (1.4) for every $\eta > 0$ is equivalent to asserting the existence of a function $\eta(n)$ such that

(2.1) $$\lim_{n\to\infty} \eta(n) = 0$$

and

(2.2) $$\lim_{n\to\infty} \sum_{k\le n} P(|X_k| > \eta(n)s_n) = 0 \quad .$$

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* $L(n)$ is defined on p. 3.

It is in the form (2.2) that the hypothesis will be applied.

Let $G(x)$ be an accumulation d.f. of the normalized partial sums S_n/s_n, and choose the corresponding subsequence $n_i \to \infty$ so that in addition for the function

$$L(n) = \frac{1}{s_n^2} \sum_{k \leq n} \int_{|x| \leq \eta(n) s_n} x^2 dF_k$$

we have

$$(2.3) \qquad L(n_i) \longrightarrow \alpha, \qquad \text{where } 0 \leq \alpha \leq 1 \quad .$$

For $k \leq n_i$ introduce

$$(2.4) \qquad X_k' = \begin{cases} X_k \text{ if } |X_k| \leq \eta(n_i) s_{n_i}; \\ 0 \text{ otherwise;} \end{cases}$$

and write

$$S_{n_i}' = X_1' + \ldots + X_{n_i}' \quad .$$

Letting μ_k' denote the mean of X_k' we have

$$(2.5) \qquad s_{n_i}'^2 = \operatorname{var}(S_{n_i}') = \sum_{k \leq n_i} \int_{|x| \leq \eta(n_i) s_{n_i}} x^2 dF_k - \sum_{k \leq n_i} \mu_k'^2 \quad .$$

For $k \leq n_i$, we have

$$|\mu_k'| = \left| \int_{|x| \leq \eta(n_i) s_{n_i}} x dF_k \right| = \left| \int_{|x| > \eta(n_i) s_{n_i}} x dF_k \right| \quad ,$$

so that by Schwarz's inequality

$$(2.6) \qquad |\mu_k'|^2 \leq P(|X_k| > \eta(n_i) s_{n_i}) \int_{|x| > \eta(n_i) s_{n_i}} x^2 dF_k \quad .$$

Summing (2.6) over all $k \le n_i$ then yields

$$\sum_{k=1}^{n_i} \mu_k'^2 \le \sum_{k=1}^{n_i} P(|X_k| > \eta(n_i)s_{n_i}) \int_{|x|>\eta(n_i)s_{n_i}} x^2 dF_k$$

$$\le \left[\sum_{k=1}^{n_i} P(|X_k| > \eta(n_i)s_{n_i})\right] s_{n_i}^2 \quad ,$$

so that

(2.7) $$\sum_{k=1}^{n_i} \mu_k'^2 = o(s_{n_i}^2) \quad .$$

Note also that we have

$$|\mu_k'| \le \sqrt{P(|X_k|>\eta(n_i)s_{n_i})} \sqrt{\int_{|x|>\eta(n_i)s_{n_i}} x^2 dF_k} \quad ,$$

so that

$$\sum_{k=1}^{n_i} |\mu_k'| \le \sqrt{\sum_{k=1}^{n_i} P(|X_k|>\eta(n_i)s_{n_i})} \sqrt{\sum_{k=1}^{n_i} \int_{|x|>\eta(n_i)s_{n_i}} x^2 dF_k}$$

$$\le o(s_{n_i}) \quad ,$$

and hence

(2.8) $$\sum_{k=1}^{n_i} \mu_k' = o(s_{n_i}) \quad .$$

From (2.3), (2.5) and (2.7) we obtain that

(2.9) $$\frac{s_{n_i}'^2}{s_{n_i}^2} \longrightarrow \alpha \text{ as } n_i \longrightarrow \infty \quad .$$

Using (2.3) again this implies

(2.10) $$\frac{1}{s_{n_i}'^2} \sum_{k\le n_i} \int_{|x|\le(\frac{\eta(n)}{\sqrt{\alpha}})s_{n_i}'} x^2 \, dF_k \longrightarrow 1 \quad ,$$

as $n_i \longrightarrow \infty$. Letting F_k' denote the d.f. of X_k', (2.10) and (2.2) yield that*

$$(2.11) \qquad \frac{1}{s_{n_i}'^2} \sum_{k \le n_i} \int_{|x| \le \left(\frac{\lambda(n)}{\sqrt{a}}\right) s_{n_i}'} x^2 \, dF_k' \longrightarrow 1 \; ,$$

as $n_i \longrightarrow \infty$. Hence from a slight extension (well known) of the Lindeberg theorem it follows that

$$(2.12) \qquad P\left(\frac{S_{n_i}' - \sum_{k=1}^{n_i} \mu_k'}{s_{n_i}'} < \omega \right) \longrightarrow \Phi(\omega) \; .$$

But from (2.8) and (2.9) this in turn implies

$$(2.13) \qquad P\left(\frac{S_{n_i}'}{s_{n_i}'} < \omega \right) \longrightarrow \Phi(\omega)$$

as $n_i \longrightarrow \infty$. Since $s_{n_i}' \sim \sqrt{a} \; s_{n_i}$ it follows that

$$(2.14) \qquad P\left(\frac{S_{n_i}'}{s_{n_i}} < \omega \right) \longrightarrow \Phi\left(\frac{\omega}{\sqrt{a}}\right) \; .$$

Finally, from (2.2)

$$P(S_{n_i}' \ne S_{n_i}) \longrightarrow 0 \text{ as } n_i \longrightarrow \infty \; ,$$

so that we obtain from (2.14) that

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* The argument as it proceeds here is valid only for $a > 0$. The validity of the final assertion in the case $a = 0$ is easily verified directly (see Example I of §3).

$$(2.15) \qquad P\left(\frac{S_{n_i}}{s_{n_i}} < \omega\right) \longrightarrow \Phi\left(\frac{\omega}{\sqrt{a}}\right)$$

as $n_i \longrightarrow \infty$, where $\Phi(\frac{\omega}{\sqrt{a}})$ is the normal distribution function with mean 0 and variance a.

Thus under the assumption of (2.2) all accumulation distribtuions, of the normalized partial sums, will be normal. Furthermore, it is easily shown that these accumulation distributions are in 1-1 correspondence with the limit points of $L(n)$, where for a such a limit point $\Phi(\frac{\omega}{\sqrt{a}})$ is the corresponding accumulation d.f. In particular, this shows that the limit points of $L(n)$ are independent of the particular choice of $\eta(x)$ used in defining $L(n)$, so long as (2.2) is satisfied.

We next proceed to the consideration of the implications of (2.2) with respect to the range of the parameter a introduced above.

<u>Lemma 2.1.</u> Assuming (2.2) we have

$$(2.16) \qquad \lim_{n \to \infty} \left\{ \frac{s_n^2}{s_{n+1}^2} L(n) - L(n+1) \right\} = 0 \quad .$$

<u>Proof</u>:

$$\frac{s_n^2}{s_{n+1}^2} L(n) = \frac{1}{s_{n+1}^2} \sum_{k \leq n} \int_{|x| \leq \eta(n) s_n} x^2 \, dF_k$$

$$= \frac{1}{s_{n+1}^2} \sum_{k \leq n+1} \int_{|x| \leq \ (n+1) s_{n+1}} x^2 \, dF_k - \frac{1}{s_{n+1}^2} \int_{|x| \leq \eta(n+1) s_{n+1}} x^2 dF_{n+1}$$

$$- \frac{1}{s_{n+1}^2} \sum_{k \leq n} \int_{(n) s_n < |x| \leq \ (n+1) s_{n+1}} x^2 \, dF_k$$

$$= L(n+1)+O(\eta^2(n+1))+O(\sum_{k\le n} P(|X_k|>\eta(n)s_n))$$

$$= L(n+1) + o(1) \quad \text{q.e.d.}$$

We are now in a position to complete the proof of the theorem, by proving

Lemma 2.2. Under the assumption of (2.2), the set of limit points of $L(n)$ fill out a single closed subinterval of $[0,1]$.

Proof: Let $\underline{c} = \underline{\lim}\, L(n)$, $\overline{c} = \overline{\lim}\, L(n)$, and consider any α, $\underline{c} < \alpha < \overline{c}$. Suppose the lemma false, so that α is not a limit point of $L(n)$. Then there is an interval about α devoid of values of $L(n)$. Suppose this interval is $[a,b]$, $b > a$. Clearly, however, there are infinitely many $L(n)$ in each of the intervals $[\underline{c},a]$, $[b,\overline{c}]$. Choose a sequence $n_i \longrightarrow \infty$ such that

$$L(n_i) \le a \quad , \quad L(n_i+1) \ge b \quad .$$

Then

$$\frac{s_{n_i}^2}{s_{n_i+1}^2} L(n_i) - L(n_i+1) \le a-b < 0 \quad ,$$

which contradicts (2.16), and completes the proof of the lemma.

We note in passing that a converse of the theorem stated in the introduction may be easily established. This converse is to the effect that if (1.2) holds and all accumulation distributions of the partial sums are normal, then (1.4) must hold.

§3. Examples. We now provide some illustrative examples which serve to establish that (1.4) does not imply the Lindeberg condition.

Example I: This will be an example in which (1.4) holds, and the parameter interval of the theorem is simply the point 0. That is, the limit distribution of the normalized partial sums is the unit function.

Choose $\lambda_1 = 2$, and define $\lambda_n > 0$ recursively via

$$(3.1) \qquad \lambda_n^2 = (n+1)^3 \sum_{k \leq n-1} \frac{k}{(k+1)^2} \lambda_k^2 .$$

Consider mutually independent random variables such that X_k has the d.f. $F_k(x)$ given by

$$(3.2) \qquad F_k(x) = \begin{cases} 0 & \text{for} \quad x \leq -\lambda_k \frac{k}{k+1} , \\ \frac{1}{k+1} & \text{for} \quad -\lambda_k \frac{k}{k+1} < x \leq \frac{\lambda_k}{k+1} , \\ 1 & \text{for} \quad x > \frac{\lambda_k}{k+1} . \end{cases}$$

Note then that

(1) the mean of F_k is $\mu_k = 0$,

(2) the variance of F_k is $\sigma_k^2 = \frac{k}{(k+1)^2} \lambda_k^2$,

(3) for $s_n^2 = \sum_{k \leq n} \sigma_k^2$ we have $s_1 = \sigma_1 = 1$ so that $s_n^2 \geq 1$.

In addition, from (3.1) we have

$$\lambda_n^2 = (n+1)^3 \, s_{n-1}^2 \geq (n+1)^3 ,$$

or

$$\lambda_n \geq (n+1)^{3/2} ,$$

so that $\lambda_n \longrightarrow \infty$ and hence also $s_n \longrightarrow \infty$. Furthermore,

$$s_n^2 = \frac{n}{(n+1)^2}\lambda_n^2 + s_{n-1}^2 = [n(n+1)+1]s_{n-1}^2$$

$$\sim n(n+1)s_{n-1}^2 = \frac{n}{(n+1)^2}\lambda_n^2 ,$$

so that

(3.3) $$\lambda_n \sim \frac{n+1}{\sqrt{n}} s_n , \qquad (\text{or } s_n \sim \frac{\sqrt{n}}{n+1}\lambda_n) .$$

(4) From (3.1) we note also that for $k \leq n-1$

$$\frac{\lambda_k k}{k+1} = \frac{\lambda_k \sqrt{k}}{k+1}\sqrt{k} \leq \frac{\lambda_n}{(n+1)^{3/2}}\sqrt{k} \sim \sqrt{\frac{k}{n}} \cdot \frac{1}{(n+1)^{1/2}} s_n = o(s_n).$$

Hence, also, uniformly for $k \leq n-1$,

$$\frac{\lambda_k}{k+1} = o(s_n) .$$

(5) $$\frac{\lambda_n}{n+1} \sim \frac{s_n}{\sqrt{n}} = o(s_n) \qquad \text{as } n \longrightarrow \infty .$$

(6) $$s_n \sim \frac{\sqrt{n}}{n+1}\lambda_n = o(\frac{n}{n+1}\lambda_n) .$$

Using (4), (5), and (6), we see that for any fixed $\eta > 0$, for n sufficiently large,

$$\sum_{k \leq n} P(|X_k| > \eta s_n) = P(|X_n| > \eta s_n) = \frac{1}{n+1} \longrightarrow 0 ,$$

as $n \longrightarrow \infty$, so that (1.4) is satisfied.

On the other hand,

$$\frac{1}{s_n^2}\sum_{k \leq n}\int_{|x|>\eta s_n} x^2\, dF_k = \frac{1}{s_n^2}\int_{|x|>\eta s_n} x^2\, dF_n$$

$$= \frac{1}{s_n^2}\cdot\frac{1}{n+1}\cdot\lambda_n^2\frac{n^2}{(n+1)^2} \sim \frac{\lambda_n^2 n}{(n+1)^2 s_n^2} \longrightarrow 1$$

as $n \longrightarrow \infty$, (by 3.3). Thus, certainly,

$$L(n) = \frac{1}{s_n^2} \sum_{k \leq n} \int_{|x| \leq \eta(n) s_n} x^2 \, dF_k \longrightarrow 0 \quad ,$$

for any $\eta = \eta(n)$ which tends to zero as $n \longrightarrow \infty$.

From this it follows easily that $\varepsilon(x)$ (the unit function) is the limit distribution of S_n/s_n. In fact for any fixed $\delta > 0$,

$$(3.4) \qquad P\left(\left|\frac{S_n}{s_n}\right| > \delta\right) \leq \sum_{k \leq n} P(|X_k| > \eta s_n) + \frac{1}{\delta_n^2} \sum_{k \leq n} \int_{|x| \leq \eta s_n} x^2 \, dF_k \quad ,$$

where the first sum on the right is a truncation error, and the second term is by Tchebychef's inequality. Thus we see that

$$P\left(\left|\frac{S_n}{s_n}\right| > \delta\right) \longrightarrow 0$$

as $n \longrightarrow \infty$, which gives the desired result.

Example II. We next give an example in which the parameter interval of the theorem is the entire closed interval [0,1]. By virtue of the theorem itself, in order to achieve this, it suffices to give an example in which both the unit function and the standard normal are accumulation distribution functions.

Let $X_1, X_2, \ldots$ be the sequence of random variables constructed in Example I, which are such that the d.f. of the normalized partial sums tends to $\varepsilon(x)$. In addition, let $Y_1, Y_2, \ldots$ be an infinite sequence of random variables such that the d.f. of the normalized partial sums tends to the standard normal, and on which the Lindeberg condition holds.* This last convergence is, as is well known, uniform.

* For convenience we take a simple case of Bernoulli variables, so that the variance of the n^{th} partial sum is n.

For $S_n = X_1 + \dots + X_n$, it follows from (3.4) that given any $\delta > 0$, there exists an $n_o = n_o(\delta)$ such that for all $n \geq n_o$, and all x such that $|x| \geq \delta$,

$$|F_n^*(x) - \varepsilon(x)| \leq \delta ,$$

where $F_n^*(x)$ is the d.f. of S_n/s_n. Recalling that we have individual negligibility, i.e. (1.2) holds, we can choose blocks of the variables of the X_k sequence

$$B_k : \{X_{t_{k-1}+1}, \dots, X_{t_k}\}$$

such that

$$(3.5) \qquad \left| \left[\text{d.f. of } \frac{X_{t_{k-1}+1} + \dots + X_{t_k}}{s_{t_k}}\right](x) - \varepsilon(x) \right| < \frac{1}{k} \quad ,$$

for $|\omega| \geq \frac{1}{k}$; (recall that $s_{t_{k-1}} = o(s_{t_k})$).

Letting $\hat{s}_n^2$ = the variance of the n^{th} partial sum of the Y_j sequence, since the convergence of the d.f.'s of the normalized partial sums to $\Phi(x)$ is uniform, we can choose blocks

$$\beta_k : \{Y_{r_{k-1}+1}, \dots, Y_{r_k}\}$$

such that

$$(3.6) \qquad \left| \left[\text{d.f. of } \frac{Y_{r_{k-1}+1} + \dots + Y_{r_k}}{s_{r_k}}\right] (x) - \Phi(x) \right| < \frac{1}{k} \quad ,$$

for all x. These blocks are also chosen so that

$$(3.7) \qquad \hat{s}_{r_{k-1}} + s_{t_k} = o(\hat{s}_{r_k}) \quad .$$

Now form the sequence of random variables consisting of both sequences $\{X_i\}$ and $\{Y_j\}$ in the order described by the blocks:

$$\{B_1, \beta_1, B_2, \beta_2, \ldots, B_k, \beta_k, \ldots\} = \{Z_1, Z_2, \ldots\} .$$

We next verify that the condition (1.4) is satisfied by the sequence $\{Z_j\}$. Let $\bar{s}_n^2$ denote the variance of the n^{th} partial sum of the Z_j; and define $k^* = k^*(n)$ to be the largest integer k such that $t_k \leq n$. Then, if Z_n is a Y_j, i.e. $Z_n = Y_\tau$ we have

$$\sum_{j \leq n} P(|Z_j| \geq \eta \bar{s}_n) \leq \sum_{i \leq t_{k^*}} P(|X_i| \geq \eta s_{t_{k^*}})$$

$$+ \sum_{\ell \leq \tau} P(|Y_\ell| \geq \eta \hat{s}_\tau) .$$

Since, as $n \longrightarrow \infty$, $t_{k^*} \longrightarrow \infty$ and $\tau \longrightarrow \infty$, both sums on the right tend to zero and (1.4) is verified in this case. A similar argument applies for the case where Z_n is an X_i.

Let $\bar{S}_{n_i}$ be the subsequence of partial sums of the Z_j corresponding to stopping at the end of the β_k blocks. Then, as a consequence of (3.7), we have

$$P\left(\frac{\bar{S}_{n_i}}{\bar{s}_{n_i}} < x\right) = P\left(\frac{Y_{r_{k-1}+1} + \ldots + Y_{r_k}}{\bar{s}_{n_i}} < x\right) + o(1) ,$$

and combining this with (3.6) yields

$$P\left(\frac{\bar{S}_{n_i}}{\bar{s}_{n_i}} < x\right) \longrightarrow \Phi(x)$$

as $n_i \longrightarrow \infty$.

Let $\bar{S}'_{n_i}$ be the subsequence of partial sums of the Z_j corresponding to stopping at the end of the B_k blocks. Then by arguments similar to those given above it is easily shown that

$$P\left(\frac{\bar{S}_{n'_i}}{\bar{s}_{n'_i}} < x\right) \longrightarrow \varepsilon(x) .$$

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